On a Theorem of Nehari and Quasidiscs

Martin Chuaqui

University of Pennsylvania

Abstract

Let f be a locally injective analytic map of the unit disc D and let $\{f,z\}$ be its Schwarzian derivative. Suppose $|\{f,z\}| \leq 2p(|z|)$. We use the classical connection between Schwarzian derivative and second order linear equations to show that, for a particular class of functions p, the image $f(D)$ is a quasidisc. The analysis centers on the differential equation $y''+py=0$ and a finiteness condition of a positive solution y. The proofs are based on Sturm comparison theorems. When p in the class is analytic and $x=1$ is a regular singular point of the linear equation, it is possible to obtain precise information about Hölder continuity of f from considerations on the Frobenius solutions at that point. The main result in this paper resolves the complementary case in a general theorem of univalence of Nehari.

1. INTRODUCTION

Let f be analytic and locally univalent, and let $\{f,z\} = (f''/f')' - (1/2)(f''/f')^2$ be its Schwarzian derivative. Two main features of the Schwarzian derivative are that all solutions to $\{f,z\}=0$ are given by fractional linear transformations $T(z)=(az+b)/(cz+d)$ and that $\{T\circ f, z\} = \{f,z\}$. The second property is a consequence of the first and an important addition formula for the Schwarzian derivative of a composition. There is a classical connection between the Schwarzian derivative and second order linear equations: any solution of $\{f,z\}=2p(z)$ is given by $(au+bv)/(cu+dv)$, $ad-bc\neq 0$, where u,v are two linearly independent solutions of the equation

$$y''+py=0\,. \tag{1.1}$$

A well known fact that follows from this is that f is univalent on a given domain if and only if any nontrivial solution of (1.1) vanishes at most once in the domain (see, e.g., [D]). This characterization of univalence was systematically used by Nehari, who derived several sufficient conditions for global injectivity. Briefly, estimates on the size of $|\{f,z\}|$ together with comparison theorems for the solutions of differential equations imply the absence of multiple zeroes of nontrivial solutions of (1.1). In the unit disc D, some of the conditions of this type that imply univalence are

$$|\{f,z\}| \leq \frac{\pi^2}{2} \tag{1.2}$$

$$|\{f,z\}| \leq \frac{2}{(1-|z|^2)^2} \tag{1.3}$$

$$|\{f,z\}| \leq \frac{4}{1-|z|^2}\,. \tag{1.4}$$

The constants $\pi^2/2$, 2 and 4 are sharp in each case. All of these are particular instances of the following general result due to Nehari (Theorem 1 in [N 2]):

Let $p(x) \geq 0$ be an even function on (-1,1) such that $(1-x^2)^2 p(x)$ is nonincreasing for $x > 0$. Suppose that the solution y of

$$y'' + py = 0\,, \quad y(0) = 1\,, y'(0) = 0 \tag{1.5}$$

does not vanish on (-1,1). If $|\{f, z\}| \leq 2p(|z|)$ then f is univalent in the unit disc.

With the choices for p as in (1.2), (1.3) and (1.4) the respective solutions of (1.5) are given by $\cos(\frac{\pi}{2}x)$, $\sqrt{1-x^2}$ and $1-x^2$. The odd solution of the linear equation is

$$y(x) \int_0^x y^{-2}(s)ds$$

and since the functions involved here are analytic on (-1,1) they extend to the unit disc. Consequently

$$F(z) = \int_0^z y^{-2}(\zeta)d\zeta$$

gives in each case the extremal map with the normalizations $F(0) = 0, F'(0) = 1$ and $F''(0) = 0$.

By using standard comparison theorems for solutions of differential equations one can go a step further and derive upper and lower bounds for $|f|$ and $|f'|$ when f is normalized as F and $|\{f, z\}| \leq 2p(|z|)$ [C-O].

In Nehari's theorem p is assumed to be continuous, and for such p we will continue to denote by F the associated function defined on (-1,1). Nehari also showed under what circumstances the condition $|\{f, z\}| \leq 2p(|z|)$ is sharp. It states that if $F(x) \to \infty$ as $x \to 1$ then for any positive function $r(x)$ on (-1,1) the condition

$$|\{f, z\}| \leq 2p(|z|) + r(|z|)$$

is in general not sufficient for univalence (Theorem 2 in [N 2]).

In this paper we shall be concerned with the question of what happens when $F(1) < \infty$. Our main result is

THEOREM 1. *Let $p(x) \geq 0$ be an even function on (-1,1) with $(1-x^2)^2 p(x)$ nonincreasing for $x > 0$. Suppose that the even solution y of (1.1) is positive and is such that*

$$\int_0^1 y^{-2}(x)dx < \infty\,.$$

If $|\{f, z\}| \leq 2p(|z|)$ then $f(D)$ is a quasidisc.

A quasidisc is the image of D under some map which is quasiconformal in the entire plane. Let Ω be simply-connected with its Poincaré metric $\lambda(z)|dz|$. A combined result of Ahlfors and later Gehring gives a characterization of quasidiscs which is necessary and sufficient: there exists a positive constant η such the inequality

$$|\{\phi, z\}| \leq \eta\lambda^2(z)$$

implies that ϕ is univalent in Ω (see, e.g., [L]).

The function p in Theorem 1 is assumed to be continuous. As previously shown, there are important cases when p is actually analytic and $x = 1$ is a regular singular point of (1.1). This allows to simplify the analysis by considering the possible Frobenius solutions at $x = 1$. The assumption that $F(1)$ is finite implies that either $y \sim (1-x)^m$ as $x \to 1$, for some $0 < m < \frac{1}{2}$, or else that $y(1) > 0$. In the latter case, the normalized function f will be Lipschitz continuous on D while in the former case, it is possible to prove Hölder continuity.

I would like to thank C. Epstein for helpful discussions concerning the proof of Lemma 1.. The referee's valuable comments allowed a simplification of the original proof and gave greater clarity to other parts of the exposition.

2. PROOFS

The proof of Theorem 1 will be divided in a series of lemmas. In what follows, let p and y satisfy the hypothesis of the theorem. Let $\alpha \in [0, 1)$. Most of the analysis ahead depends on the solution u of

$$u'' + \frac{\alpha}{(1-x^2)^2} u = 0, \quad u(0) = 1\,, u'(0) = 0\,. \tag{2.1}$$

This function is given explicitly by

$$u(x) = \frac{1}{2}\sqrt{1-x^2}\left\{\left(\frac{1+x}{1-x}\right)^\beta + \left(\frac{1-x}{1+x}\right)^\beta\right\}$$

where $\beta = \frac{1}{2}\sqrt{1-\alpha}$ [K, p.492]. In particular,

$$u(x) \sim (1-x)^{\frac{1}{2}-\beta} \quad , x \to 1$$

and therefore $\int_0^1 u^{-2}(x)dx < \infty$. Let $\mu = \lim_{x\to 1}(1-x^2)^2 p(x)$. Clearly $\mu \geq 0$ and we claim that $\mu < 1$. If not then $p(x) \geq (1-x^2)^{-2}$. Let $P(x) = (1-x^2)^{-2}$ so that the function $q(x) = p(x) - P(x)$ is non-negative. Then $z(x) = \sqrt{1-x^2}$ satisfies

$$z'' + Pz = 0, \quad z(0) = 1\,, z'(0) = 0\,. \tag{2.2}$$

Multiplying (2.2) by y, (1.5) by z, and subtracting, we get

$$z''y - zy'' = qyz\,.$$

We integrate this equation, using the initial condition on y and z, to obtain

$$(\frac{z}{y})'(x) = \frac{\int_0^x (uqy)(s)ds}{y(x)^2}\,.$$

Hence $(\frac{z}{y})'$ has the same sign as x and therefore $y \leq z$ on (-1,1) since $z(0) = y(0) = 1$. It follows that either y vanishes on (-1,1) or else $F(1) = \infty$. This contradiction proves our claim. Choose α such that $\mu < \alpha < 1$ and let now

$$q(x) = p(x) - \frac{\alpha}{(1-x^2)^2}\,.$$

LEMMA 1. *Let* $l = \liminf_{x\to 1}(1-x)(y'/y)$. *Then* $-\frac{1}{2} < l \leq 0$.

PROOF: Following an argument almost identical to the one given above, we can write

$$\frac{y'}{y} = \frac{u'}{u} - \frac{\int_0^x (uqy)(s)ds}{u(x)y(x)}\,. \tag{2.3}$$

Since $y'' = -py \leq 0$ we have $y' \leq 0$ on (0,1) because of the initial condition. Hence $l \leq 0$. On the other hand, the limit of $(1-x)(u'/u)$ as $x \to 1$ can be computed directly and it equals $-(\frac{1}{2} - \beta)$. This, together with equation (2.3) and the fact that $q(x) < 0$ for x sufficiently close to 1 imply the lemma.

By considering the graph of the function F it follows from elementary geometry that

$$\lim_{x\to 1} \frac{1-x}{y^2} = 0\,.$$

LEMMA 2. *There exists a constant* M *such that*

$$F(1) - F(x) \leq M\left(\frac{1-x}{y^2}\right)\,. \tag{2.4}$$

PROOF: The derivative of the left hand side of (2.4) is $-y^{-2}$ while the derivative of $(1-x)y^{-2}$ is

$$-y^{-2}(1 + 2(1-x)\frac{y'}{y})\,.$$

Lemma 1 implies that $1 + 2(1-x)(y'/y) \geq \sigma > 0$ provided x is sufficiently close to 1. Hence for all such x

$$F(1) - F(x) \leq \frac{1}{\sigma}\left(\frac{1-x}{y^2}\right)$$

and the lemma follows.

Now we state the key result in this chain.

LEMMA 3. *There exists a constant $\eta > 0$ such that the solution φ of*

$$\varphi'' + (p(x) + \frac{\eta}{(1-x^2)^2})\varphi = 0, \quad \varphi(0) = 1\,, \varphi'(0) = 0 \tag{2.5}$$

does not vanish on (-1,1).

PROOF: Let $c = F(1)$. On the image interval $(-c, c)$ we consider the "Poincaré density"

$$\lambda(w) = \frac{1}{F'(x)(1-x^2)} = \frac{y^2}{1-x^2}$$

where $w = F(x)$. We will show that for $\eta > 0$ sufficiently small the solution h of

$$h'' + \eta\lambda^2(w)h = 0, \quad h(0) = 1\,, h'(0) = 0 \tag{2.6}$$

is positive on $(-c, c)$. By Lemma 2,

$$\lambda(w) = \frac{y^2}{1-x^2} \leq \frac{y^2}{1-x} \leq \frac{M}{c-w} \leq \frac{2Mc}{c^2-w^2}\,.$$

Thus it suffices to show that the solution of (2.6) with $\lambda^2(w)$ replaced by $4M^2c^2(c^2-w^2)^{-2}$ does not vanish. This will be the case as long as $4M^2\eta \leq 1$. To see this, we rescale. The function $\bar{h}(x) = h(cx)$ solves $\bar{h}'' + 4M^2\eta(1-x^2)^{-2}\bar{h} = 0$ with even initial conditions. Then $\bar{h} > 0$ on (-1,1) if and only if $4M^2\eta \leq 1$ [K, p.492]. With h the positive solution of (2.6) we define φ by

$$\varphi(x) = y(x)h(F(x))\,.$$

A straightforward computation shows that φ is the solution of (2.5). This finishes the proof of Lemma 3.

This lemma together with Nehari's first theorem shows that

$$|\{g, z\}| \leq 2(p(|z|) + \frac{\eta}{(1-|z|^2)^2}) \tag{2.7}$$

is a sufficient condition for univalence. The proof of Theorem 1 is now quite simple. Assume $|\{f, z\}| \leq 2p(|z|)$ and let $\lambda(\zeta)|d\zeta|$ be the Poincaré metric on $\Omega = f(D)$. We will show that

$$|\{\phi, \zeta\}| \leq 2\eta\lambda^2(\zeta)$$

implies the univalence of the map ϕ. Let $g(z) = \phi(f(z))$. Then

$$\{g, z\} = \{\phi, f(z)\}f'(z)^2 + \{f, z\}$$

and therefore

$$(1-|z|^2)^2|\{g, z\}| \leq \lambda^{-2}(\zeta)|\{\phi, \zeta\}| + 2(1-|z|^2)^2 p(|z|)$$

where $\zeta = f(z)$. It follows that g satisfies (2.7), hence g and consequently ϕ are univalent. This shows that Ω is a quasidisc.

3. THE ANALYTIC CASE

In this section we shall assume that, in addition, p is analytic and that $x = 1$ is a regular singular point of the equation (1.1). The assumptions on the even solution y are as before. Recall that $\mu = \lim_{x\to 1}(1-x^2)^2 p(x)$. From the analysis of the possible Frobenius solutions at $x = 1$ we will prove Hölder or Lipschitz continuity for maps f that satisfy $|\{f, z\}| \leq 2p(|z|)$. Because of the invariance of the Schwarzian derivative under Möbius changes one can not expect such a result unless f is properly normalized. The right normalization turns out to be $f''(0) = 0$. Let u solve

$$u'' + \frac{1}{2}\{f, z\}u = 0, \quad u(0) = 1\,, u'(0) = 0$$

and let

$$v(z) = u(z)\int_0^z u^{-2}(\zeta)d\zeta$$

be the solution with odd initial conditions. If $f''(0) = 0$ then

$$f(z) = f(0) + f'(0)\int_0^z u^{-2}(\zeta)d\zeta\,.$$

From Lemma 2 in [C-O] it follows that if $|\{f, z\}| \leq 2p(|z|)$ then

$$|u(z)| \geq y(|z|)$$

and therefore

$$|f'(z)| \leq |f'(0)|y^{-2}(|z|)\,.$$

We distinguish the cases $\mu > 0$ and $\mu = 0$. Suppose μ is positive. Then $p(x) \geq \mu(1-x^2)^{-2}$ and hence the function y must vanish at $x = 1$. The possible orders m of vanishing are given by the roots of the inditial equation

$$m^2 - m + \frac{\mu}{4} = 0\,,$$

i.e.,

$$m_1 = \frac{1+\sqrt{1-\mu}}{2} \quad , \quad m_2 = \frac{1-\sqrt{1-\mu}}{2}\,. \tag{3.1}$$

Note that $0 < m_2 < \frac{1}{2} < m_1 < 1$. Since $m_1 - m_2$ is not an integer both orders of vanishing can occur [H].

THEOREM 2. *Let f satisfy $|\{f, z\}| \leq 2p(|z|)$, $f''(0) = 0$ and suppose $\mu > 0$. If $F(1)$ is finite then f is Hölder continuous on D with Hölder exponent $\sqrt{1-\mu}$.*

PROOF: The assumption that $F(1) < \infty$ implies that $y \sim (1-x)^{m_2}$ as $x \to 1$. Therefore

$$|f'(z)| = O((1-|z|)^{-2m_2})\,.$$

A standard technique of integrating along hyperbolic segments (see, e.g., [G-P]) gives

$$|f(z_1) - f(z_2)| = O(|z_1 - z_2|^{1-2m_2})\,,$$

and the theorem follows.

Suppose now $\mu = 0$. In this case, the roots of the inditial equation are 1 and 0. Hence two linearly independent solutions are $y_1 = (1-x)h_1$ and $y_2 = h_2 + cy_1 \log(1-x)$, where h_1, h_2 are analytic and nonvanishing at $x = 1$ [H, Theorem 5.3.1].

THEOREM 3. *Let f satisfy $|\{f,z\}| \le 2p(|z|)$, $f''(0) = 0$ and suppose $\mu = 0$. If $F(1)$ is finite then f is Lipschitz continuous on D.*

PROOF: The finiteness condition and the discussion preceding the theorem imply that in fact y cannot vanish at $x = 1$. Hence $|f'|$ is uniformly bounded.

The following situation describes accurately the case $\mu = 0$. Let $p(x) = 2(1-x^2)^{-1}$ and let $p_t(x) = tp(x)$, $0 \le t < 1$. Since the inequality $|\{f,z\}| \le 2p(|z|)$ is sufficient for univalence then $|\{f,z\}| \le 2p_t(|z|)$ implies that $f(D)$ is a quasidisc (Theorem 6 in [G-P]). As mentioned in the introduction, the even solution of (1.1) is in this case $y = 1 - x^2$. We claim that the even solution y_t of (1.1) with p replaced by p_t must be positive at the endpoints. To show this, let

$$F_t(z) = \int_0^z y_t^{-2}(\zeta)d\zeta\,.$$

This function is odd and has Schwarzian derivative equal to $2p_t(z)$. Therefore $F_t(D)$ is a quasidisc and hence $F_t(1) < \infty$, otherwise the point at infinity would be a point of self-intersection of $\partial F_t(D)$. This in turn would contradict the fact that $\partial F_t(D)$ is a Jordan curve. Since $\mu_t = t\mu = 0$ it follows that y_t is a linear combination of the functions y_1, y_2 as in the paragraph preceding the statement of Theorem 3.. Thus $F_t(1) < \infty$ forces $y_t(1) > 0$.

We consider finally examples for any $\mu \in (0,1)$. For $s \in (1,2)$ let

$$p(x) = s\frac{1-(s-1)x^2}{(1-x^2)^2}\,.$$

Then $\mu = s(2-s)$ and the even solution of (1.1) is

$$y = (1-x^2)^{\frac{s}{2}}\,.$$

(The exponent $\frac{s}{2}$ corresponds to m_1 in (3.1) and $m_2 = \frac{2-s}{2}$.) This shows that the function F has $F(1) = \infty$. On the other hand, by the argument given above, changing p to tp has the effect of making $F_t(1)$ finite. Consider now equation (3.1) with μ replaced by $t\mu$. Since $y_t \sim (1-x)^{m_1}$, $x \to 1$, would make $F_t(1)$ infinite, we conclude that the order of vanishing of the solution y_t must be the other root, m_2.

REFERENCES

[A-H] J.M. Anderson and A. Hinkkanen, *Positive superharmonic functions and the Hölder continuity of conformal mappings*, J. London Math. Soc. 39 (1989), 256-270

[C-O] M. Chuaqui and B. Osgood, *Sharp distortion theorems associated with the Schwarzian derivative*, preprint, 1991

[D] P. Duren, *Univalent Functions*, Springer Verlag, New York, 1983

[G-P] F. Gehring and C. Pommerenke, *On the Nehari univalence criterion and quasicircles*, Comment. Math. Helv. 59 (1984), 226-242

[H] E. Hille, *Ordinary Differential Equations in the Complex Domain*, John Wiley and Sons, New York, 1976

[K] E. Kamke, *Differentialgleichungen*, Chelsea, New York, 1948

[L] O. Lehto, *Univalent Functions and Teichmüller Spaces*, Springer Verlag, New York, 1987

[N 1] Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. v.55 (1949), 545-551

[N 2] ———, *Univalence criteria depending on the Schwarzian derivative*, Illinois J. of Math. v.23 (1979), 345-351

Mathematics Subject Classification:

30C62: Quasiconformal Mappings in the Plane
34C11: Growth, boundedness and comparison of solutions of ODE's